Non-Perturbative Schwinger-Dyson Equations for 3d N=4 Gauge Theories

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In Physics, a correlator is defined in quantum field theory via a path integral, with some contour prescription. It is physically desirable to ask that the path integral be invariant under an infinitesimal shift of the contour (provided that the integral measure is left invariant by such a shift). Schwinger and Dyson explained that this constraint can be recast as a set of equations satisfied by such a correlator, and that the equations will encode a symmetry of the underlying quantum field theory.

For example, consider the path integral of four-dimensional Yang-Mills theory, with its contour defined in some topological sector, and shift the contour to a different topological sector, related to the former by a large gauge transformation. Recall that the connected components of the space of gauge fields are labeled by an integer called the instanton charge: $\frac{-1}{8\pi^2}\int_{\mathbb{C}^2}\mathrm{Tr}(F\wedge F)$, where F is the field strength and the domain of integration is the spacetime. Put differently: what are the Schwinger-Dyson equations associated with varying the instanton charge? What are the corresponding symmetries of Yang-Mills theory?

Thanks to the success of equivariant localization methods, this question was recently answered in the context of 4d $\mathcal{N} = 2$ supersymmetric Yang-Mills, on a regularized spacetime called the Ω-background on $\mathbb{C}^2$.
[Nekrasov '02] [Losev-Marshakov-Nekrasov '03] [Nekrasov-Witten '10]

The idea is as follows: on this background, the instanton number can be changed by adding and removing point-like instantons in a controlled way, and the shift of contour in the definition of the path integral turns into the discrete operation of adding and removing boxes in a Young tableau [Nekrasov-Okounkov '03].

One finds that the Schwinger-Dyson equations encode a Yangian symmetry of the theory [Nekrasov '15]. More precisely, the equations are a set of regularity conditions on the twisted (deformed) character of some finite dimensional irreducible representation of a Yangian algebra.

The above result can be generalized in many ways, and has been an active subject of investigation:

- Adding surface defects. [Nekrasov ’17] [Jeong-Nekrasov ’18]
- Yang-Mills with different gauge groups. [Haouzi-Oh ’20]
- K-theoretic instanton counting (5d Super Yang-Mills on a circle) leads to quantum affine symmetry. [Kimura-Pestun ’15] [Kim ’16] [Mironov-Morozov-Zenkevich ’16] [Chang-Ganor-Oh ’16] [Kimura-Mori-Sugimoto ’17] [Bourgine-Fukada-Harada-Matsuo-Zhu ’17] [Assel- Sciarappa ’18] [Haouzi-Kozçaz ’19] [Bourgine-Jeong ’19] [Haouzi ‘20]
- Elliptic cohomology (6d Super Yang-Mills on a 2-torus) leads to quantum elliptic symmetry. [Kimura-Pestun ’16] [Agarwal-Kim-Kim-Sciarappa ’18]

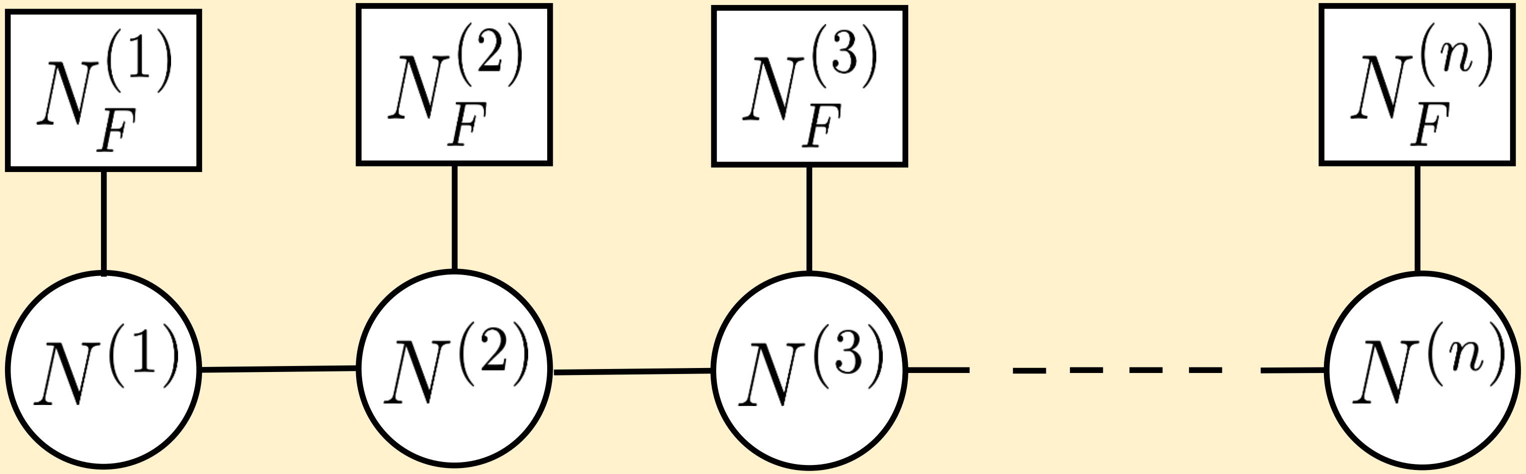
Remarkably, equivariant localization can be performed to yield exact expressions and non-perturbative Schwinger-Dyson identities in all of the above cases.

What about lower-dimensional gauge theories? For example, consider a two-dimensional gauged linear sigma model on the complex line $\mathbb{C}$. There exists once again distinct topological sectors of the theory, this time labeled by an integer called the vortex charge: $\frac{-1}{2\pi}\int_{\mathbb{C}}\mathrm{Tr}F$, where F is the field strength and the integration is over the complex line transverse to the vortex. Then, it is natural to ask what the associated Schwinger-Dyson equations are in this case [Nekrasov '17].

In this talk, we address this question, again in a supersymmetric setting.

In fact, it will be a fruitful endeavor to work in a K-theoretic framework: For definiteness, let G^{3d} be a 3d $\mathcal{N}=4$ quiver gauge theory defined on the manifold $\mathbb{C}\times S^1(\widehat{R})$, where the quiver is labeled by a simply-laced Lie algebra $\mathfrak{g}$ of rank n. The Lagrangian is captured by the following quiver diagram, of shape the Dynkin diagram of $\mathfrak{g}$, where positive integers label the rank of unitary gauge group $G=\prod_{a=1}^{n} U(N^{(a)})$ and flavor symmetry group $G_F=\prod_{a=1}^{n} U(N_F^{(a)})$:

$\text{3d Quiver } G^{3d}$:



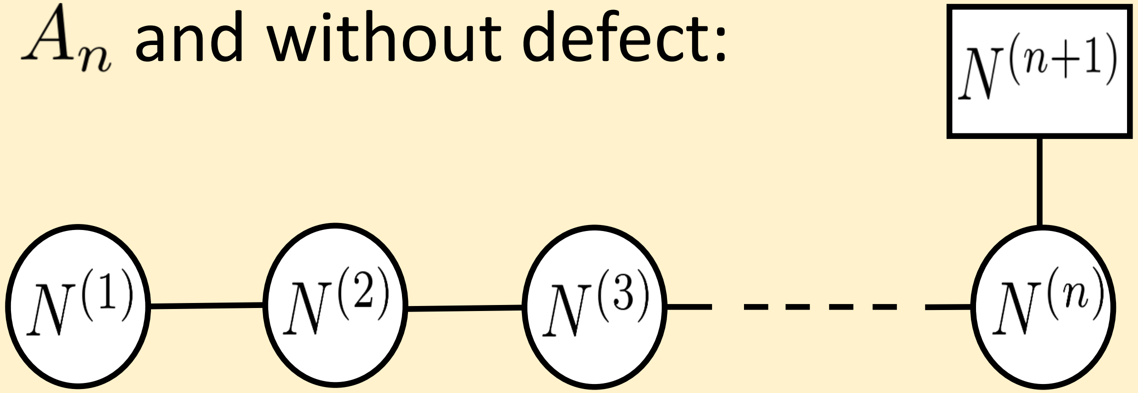
The theory G^{3d} has a moduli space of vacua called Coulomb and Higgs branches, and a corresponding $SU(2)_C \times SU(2)_H$ R-symmetry, where each $SU(2)$ acts on the two branches separately.
Our goal is to exhibit certain symmetries associated to finite energy configurations of BPS vortices, which sit at Higgs vacua of G^{3d}. Therefore, from now on, we require that G3d possess a Higgs branch (i.e. existence of a Nakajima quiver variety), and moreover that all vacua we study be Higgs vacua. In other words, the flavor symmetry group G_F should have a large enough rank. The vortices we study are semi-local non-abelian versions of [Nielsen-Olesen '73] solutions.

In this discussion, the main actor in this talk is a codimension-2 line defect of G^{3d} at a point on $\mathbb{C}$ and wrapping $S^1(\widehat{R})$, which will mediate the change in vortex number. The cleanest way to describe the physics is to realize that both the vortices and the line defect are particles in three dimensions. Then, let T^{1d} be the one-dimensional gauged quantum mechanics living on the vortices of G^{3d}, in the presence of the defect.

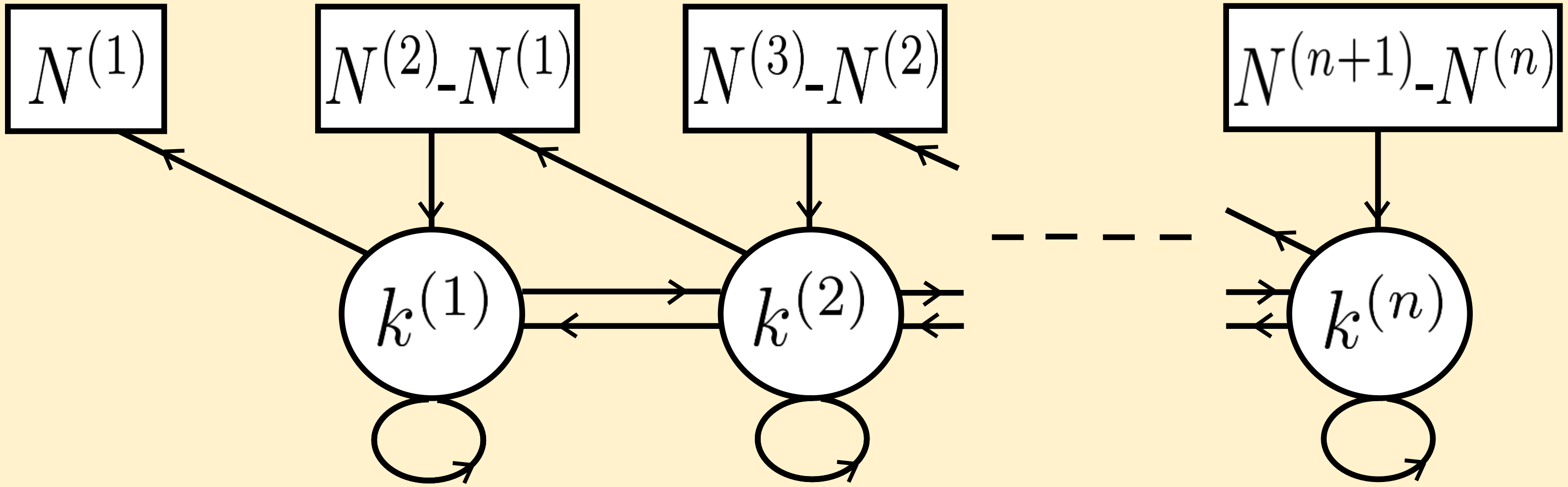
T^{1d} arises as follows: On the Higgs branch of G^{3d}, the gauge group G breaks to its $U(1)$ centers. Correspondingly, we turn on n F.I. parameters, which breaks the R-symmetry to $SU(2)_C \times U(1)_H$, and study the resulting 1/2-BPS vortex solutions. The resulting quantum mechanics has 1d $\mathcal{N} = 4$ supersymmetry, with supercharges inherited from the 3d theory. It is again of quiver type, with gauge group given by the vortex numbers, $\prod_{a=1}^{n} U(k^{(a)})$, and flavor group determined from the 3d theory masses.

As an example, when $\mathfrak{g} = A_n$ and without defect:

3d Quiver G^{3d} :

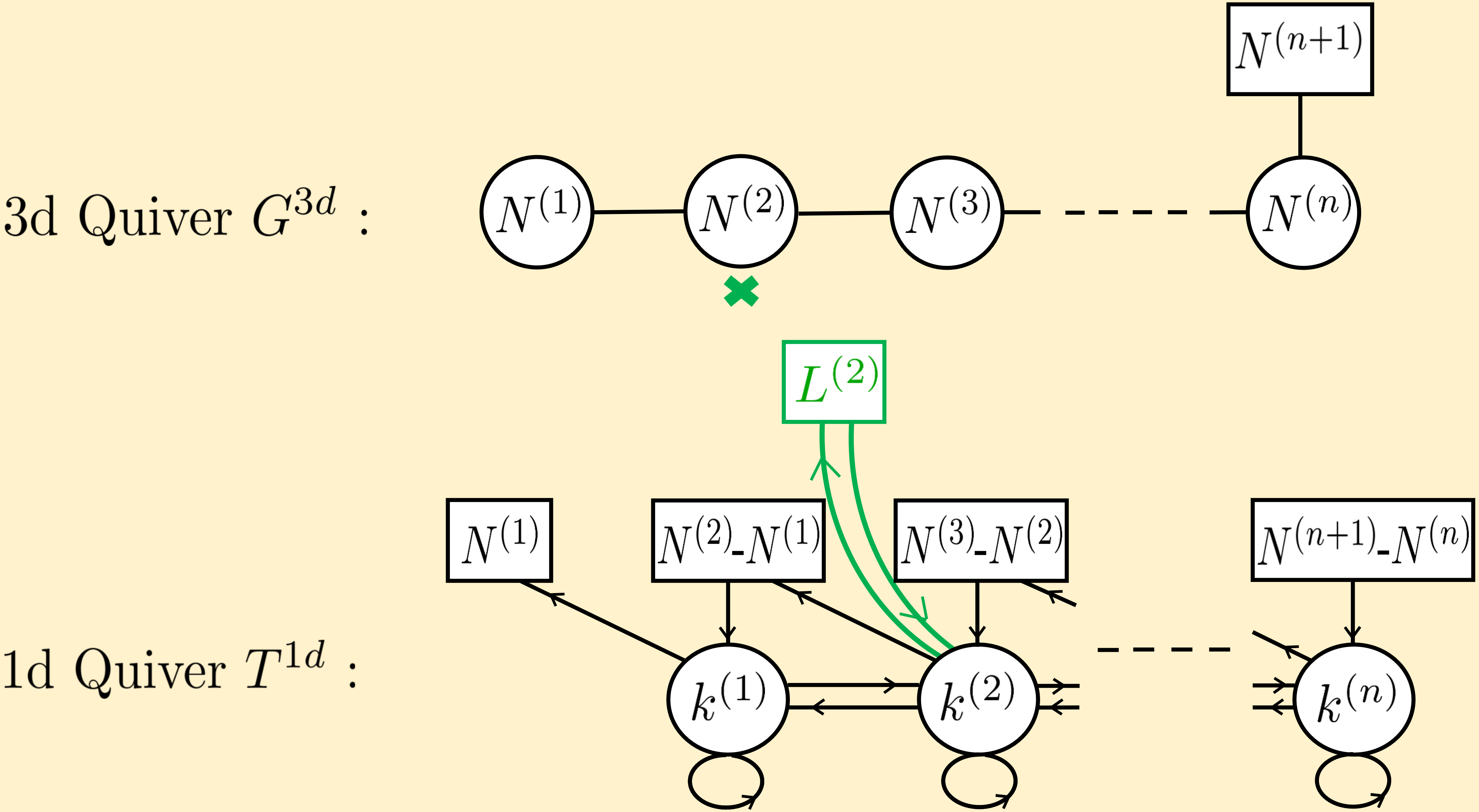


1d Quiver T^{1d}_{pure} :



T^{1d}_{pure} is sometimes called a 'hand-saw' quiver, isomorphic to a parabolic Laumon space [Finkelberg-Rybnikov '10]. It is the moduli space of based quasi maps from $\mathbb{P}^1$ into a flag variety [Venugopalan-Woodward '13] .

The data characterizing the line defect is an extra flavor group $\prod_{a=1}^{n} U(L^{(a)})$. Additional 1d $\mathcal{N}=4$ chiral matter transforms in the fundamental representation of that group. In our previous example, this reads:



We write the partition function of T^{1d} as the (twisted) Witten index of the following $\mathcal{N}=2$ gauged quantum mechanics:

$$Z = \mathrm{Tr}_{\mathcal{H}_{\mathrm{QM}}}\left[(-1)^F\, e^{\widehat{R}\,\epsilon_2(2J_3-r)}\, e^{-\widehat{R}\{Q,\overline{Q}\}}\,\mathfrak{m}\right]$$

The trace is taken over all BPS states of the quantum mechanics. The index counts states in Q-cohomology. We denoted the fermion number by F, J_3 is a generator of the Cartan subalgebra of $SU(2)_C$, while r is a generator of $U(1)_H$. $\widehat{R}$ is the radius of the circle, and $\mathfrak{m}$ stands collectively for all the twists by the flavor symmetries of the quantum mechanics:

$$\mathfrak{m} = e^{2\widehat{R}\,\epsilon_1 J_-}\prod_{a=1}^{n} e^{\widehat{R}\sum_d m_d^{(a)}\Pi_d^{(a)}}\; e^{\widehat{R}\sum_\rho M_\rho^{(a)}\Lambda_\rho^{(a)}}$$

The index is the grand canonical ensemble of vortex BPS states. There is a natural grading by the vortex numbers $k^{(a)} = \frac{-1}{2\pi} \int_{\mathbb{C}} \mathrm{Tr} F^{(a)}$, which are the topological $U(1)$ charges conjugate to the F.I. parameters $\zeta_{3d}^{(a)}$ in 3d, and the rank of the gauge group in 1d.

Then, the quantum mechanics index can be organized as a sum over the vortex sectors $(k^{(1)}, k^{(2)}, \ldots, k^{(n)})$.

The Witten index does not depend on the circle scale $\widehat{R}$. In particular, we can work in the limit $\widehat{R} \to 0$, where it reduces to Gaussian integrals around saddle points. These saddle points are parameterized by $\phi^{(a)} = \widehat{R}\,\varphi^{(a)}_{1d} + i\,\widehat{R}\,A^{(a)}_{t,1d}$, the scalar and the gauge field in the vector multiplet of the quantum mechanics. We denote the gauge group of this quantum mechanics as $\widehat{G}$, and the (complexified) eigenvalues of $\phi^{(a)}$ as $\phi^{(a)}_1, \dots, \phi^{(a)}_{k^{(a)}}$.

Performing the Gaussian integrals over massive fluctuations, the index reduces to a zero mode integral of various 1-loop determinants, which we write schematically as:

$$[\chi]^{(L^{(1)},\dots,L^{(n)})}_{1d} = \sum_{k^{(1)},\dots,k^{(n)}=0}^{\infty} \prod_{a=1}^{n} \frac{e^{\widehat{R}\,\zeta^{(a)}_{3d}\,k^{(a)}}}{k^{(a)}!} \times$$

$$\times \oint \left[\frac{d\phi^{(a)}_I}{2\pi i}\right] Z^{(a)}_{pure,vec} \cdot Z^{(a)}_{pure,adj} \cdot Z^{(a)}_{pure,teeth} \cdot \prod_{b\neq a}^{n} Z^{(a,b)}_{pure,bif} \cdot \prod_{\rho=1}^{L^{(a)}} Z^{(a)}_{defect}$$

Crucially, the Witten index also depends implicitly on additional continuous parameters in a piecewise constant manner: the n F.I. parameters $\zeta_{1d}^{(a)}$, which are themselves $k^{(a)}$-vectors, one for each abelian factor in gauge group of T^{1d}. Indeed, when such a parameter changes sign and crosses the value $\zeta_{1d}^{(a)} = 0$, a non-compact Coulomb branch opens up, and some vacua may appear or disappear, resulting in wall crossing and a jump in the index. This dependence on the 1d F.I. parameters is in one-to-one correspondence with the choice of the index integration contours. For the purpose of this talk, it is enough to claim that such a contour prescription exists, and goes by the name of Jeffrey-Kirwan (JK) residue prescription [Jeffrey-Kirwan '93].

[Hwang-Kim-Kim-Park '14] [Cordova-Shao '14] [Hori-Kim-Yi '14]

We choose to work in the chamber $\zeta_{1d}^{(a)} > 0$ for all a. Other chambers realize 3d Seiberg-dual theories…

We now argue that we can make contact with the representation theory of quantum affine algebras by splitting the choice of contours for the index in a clever way. Namely, we organize the poles into two sets: for a given vortex number $k = \sum_{a=1}^{n} k^{(a)}$, let $\mathcal{M}_k$ be the set of poles selected by the JK-residue prescription in the Witten index. Meanwhile, let $\mathcal{M}_k^{pure}$ be the set poles selected by the JK-residue prescription for the “pure” Witten index, in the absence of the loop defect.

For a given vortex number k, the set $\mathcal{M}_k$ is strictly larger than the set $\mathcal{M}_k^{pure}$, since the defect loop factor $Z_{defect}^{(a)}$ always contains JK-poles depending on $M_\rho^{(a)}$:

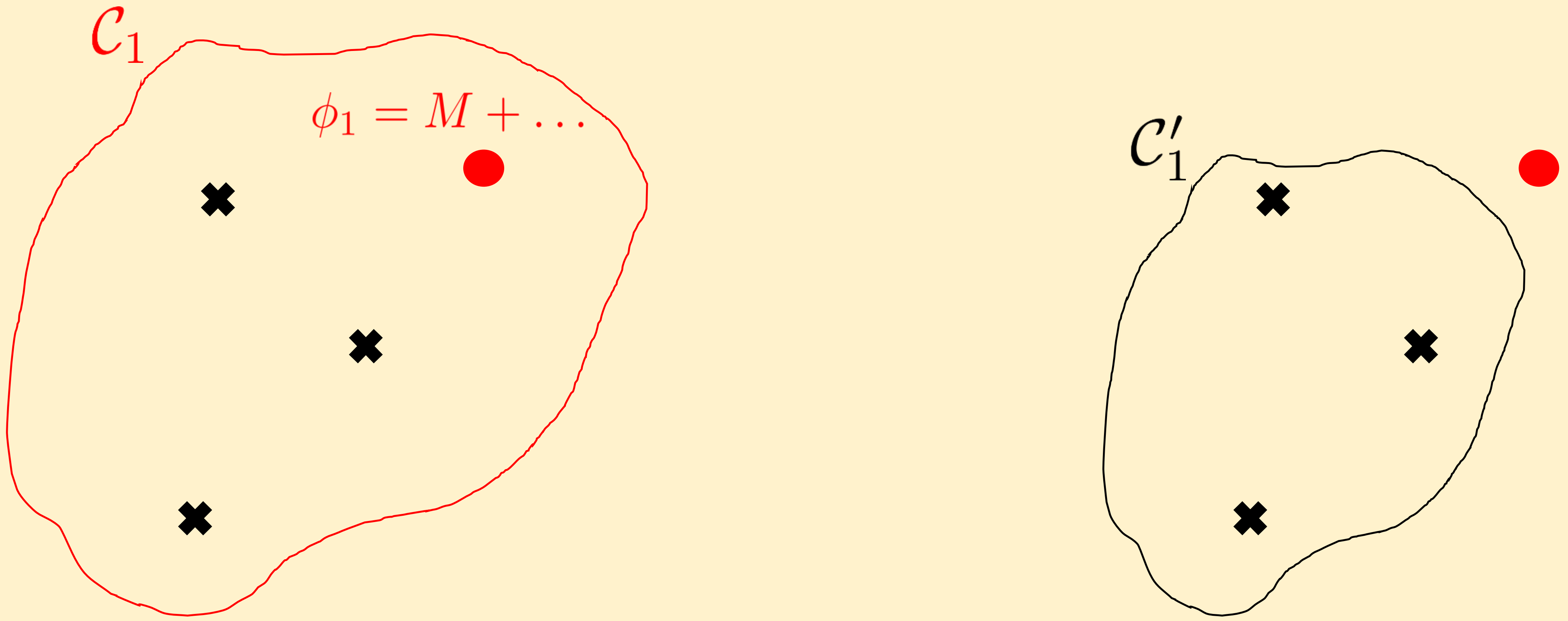
$$|\mathcal{M}_k| > |\mathcal{M}_k^{pure}|$$

We then define a new observable, the vacuum expectation value of a ``loop defect operator" on node a, with corresponding mass $M_\rho^{(a)} \equiv M$:

$$\left\langle \left[Y_{1d}^{(a)}(M) \right]^{\pm 1} \right\rangle \equiv \sum_{k^{(1)},\ldots,k^{(n)}=0}^{\infty} \prod_{b=1}^{n} \frac{e^{\widehat{R}\,\zeta_{3d}^{(b)}\,k^{(b)}}}{k^{(b)}!} \times$$

$$\times \oint_{\mathcal{M}_k^{pure}} \left[\frac{d\phi_I^{(b)}}{2\pi i} \right] Z_{pure,vec}^{(b)} \cdot Z_{pure,adj}^{(b)} \cdot Z_{pure,teeth}^{(b)} \cdot \prod_{c\neq b}^{n} Z_{pure,bif}^{(b,c)} \cdot \left[Z_{defect}^{(a)}(M) \right]^{\pm 1}$$

Note the contour definition.



Example of 1-vortex contour for $G = SU(3)$. The black crosses are poles in the set $\mathcal{M}_1^{pure}$, while the red dot is a pole in the set $\mathcal{M}_1 \setminus \mathcal{M}_1^{pure}$. The left contour is used in defining the Witten index, while the right contour is used in defining the defect Y-operator vev.

Our first result is the following: the Witten index of T^{1d} can be presented as:

$$[\chi]_{1d}^{(L^{(1)},\dots,L^{(n)})} = \left\langle \prod_{a=1}^{n} \prod_{\rho=1}^{L^{(a)}} Y^{(a)}(M_\rho^{(a)}) \right\rangle + \dots$$

The index has the same integrand as the defect operator vev, which is the first term on the right-hand side, $\left\langle \prod_{a=1}^{n} \prod_{\rho=1}^{L^{(a)}} Y^{(a)}(M_\rho^{(a)}) \right\rangle$. However, the contours on the left-hand side enclose more poles than those of the first term, since $|\mathcal{M}_k| > |\mathcal{M}_k^{pure}|$, for each vortex number k. The dots are there to make up for that deficit of $M_\rho^{(a)}$-poles, and result in a finite number of terms.

We emphasize here that at no point in the discussion do we need to know how to compute the index of the vortex theory in the absence of the defect. In other words, we do not need to know the content of the set of poles $\mathcal{M}_k^{pure}$. Instead, what is important here is the set of poles $\mathcal{M}_k \setminus \mathcal{M}_k^{pure}$.

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We find that the Witten index is a Laurent polynomial of $(1 + |\mathcal{M}_k \setminus \mathcal{M}_k^{pure}|)$ terms in Y-operator vevs (and possibly derivatives thereof). It is a deformed twisted character of the finite-dimensional representation of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, with highest weight the first term $\prod_{a=1}^{n} \prod_{\rho=1}^{L^{(a)}} Y^{(a)}(M_\rho^{(a)})$.
The character is refined by the presence of flavor and R-symmetry fugacities ϵ_1 and ϵ_2 , so we call it a vortex qq-character.
Note that If we take the size of the circle to zero in the 3d theory, we obtain a 2d (4,4) gauged sigma model with a ½-BPS point defect at the origin, and the above becomes the vortex qq-character of a Yangian algebra instead, with the same functional form as above.

Our first result is the following: the Witten index of T^{1d} can be presented as:

$$[\chi]_{1d}^{(L^{(1)},\dots,L^{(n)})} = \left\langle \prod_{a=1}^{n} \prod_{\rho=1}^{L^{(a)}} Y^{(a)}(M_\rho^{(a)}) \right\rangle + \dots$$

The Schwinger-Dyson identities are a statement about the regularity behavior of the Witten index in the fugacities $M_\rho^{(a)}$. Namely, the vev $\left\langle \prod_{a=1}^{n} \prod_{\rho=1}^{L^{(a)}} Y^{(a)}(M_\rho^{(a)}) \right\rangle$ has many apparent singularities in $M_\rho^{(a)}$, but the quantum affine symmetry of the theory eliminates all but a finite number of them, thanks to the dotted terms.

A brief guide to the literature on quantum affine algebras:

The initial construction is due to [Jimbo '86] and [Drinfeld '87].

The systematic study of the representation theory of these algebras was initiated by [Chari-Pressley '94].

Characters of finite-dimensional irreducible representations, dubbed q-characters, were first constructed by [Frenkel-Reshetikhin '98] in the 90's.

They were later rediscovered in a physical context when discussing K-theoretic instanton counting, i.e. the quantum geometry of 5d supersymmetric quiver gauge theories on a circle [Nekrasov-Pestun-Shatashvili '13] [Bullimore-Kim-Koroteev '14].

A deformed character depending on two parameters was introduced in [Awata-Harunobu-Odake-Shiraishi '95] [Awata-Harunobu-Odake-Shiraishi '95] [Frenkel-Reshetikhin '97] (see also the work of [Nakajima '00] on t-analogues of q-characters).

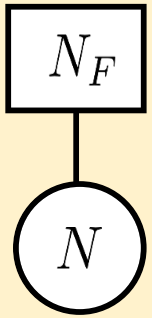
This "qq-character" was again rediscovered in the context of 5d supersymmetric gauge theories [Nekrasov '15].

Here, we see that such deformed characters also appear in the context of vortex counting in 3d gauge theories.

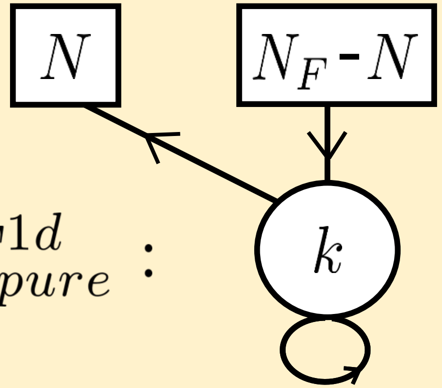
Example: SQCD, first without defect:

$(\mathfrak{g} = A_1)$

3d Quiver G^{3d} :



1d Quiver T^{1d}_{pure} :



$$[\chi]^{(0)}_{1d} = \sum_{k=0}^{\infty} \frac{e^{\zeta_{3d}\, k}}{k!} \oint_{\mathcal{M}_k^{pure}} \left[\frac{d\phi_I}{2\pi i}\right] Z_{pure,vec} \cdot Z_{pure,adj} \cdot Z_{pure,teeth} \ ,$$

$$Z_{pure,vec} = \frac{\prod_{\substack{I \neq J \\ I,J=1}}^{k} \mathrm{sh}\left(\phi_I - \phi_J\right)}{\prod_{I,J=1}^{k} \mathrm{sh}\left(\phi_I - \phi_J + \epsilon_2\right)}$$

$$Z_{pure,adj} = \prod_{I,J=1}^{k} \frac{\mathrm{sh}\left(\phi_I - \phi_J + \epsilon_1 + \epsilon_2\right)}{\mathrm{sh}\left(\phi_I - \phi_J + \epsilon_1\right)}$$

$$Z_{pure,teeth} = \prod_{I=1}^{k} \prod_{i=1}^{N} \frac{\mathrm{sh}\left(\phi_I - m_i + (\epsilon_1 + \epsilon_2)/2\right)}{\mathrm{sh}\left(\phi_I - m_i + (\epsilon_1 - \epsilon_2)/2\right)} \prod_{j=N+1}^{N_f} \frac{\mathrm{sh}\left(-\phi_I + m_j + (\epsilon_1 + 3\epsilon_2)/2\right)}{\mathrm{sh}\left(-\phi_I + m_j + (\epsilon_1 + \epsilon_2)/2\right)}$$

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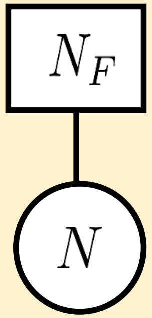

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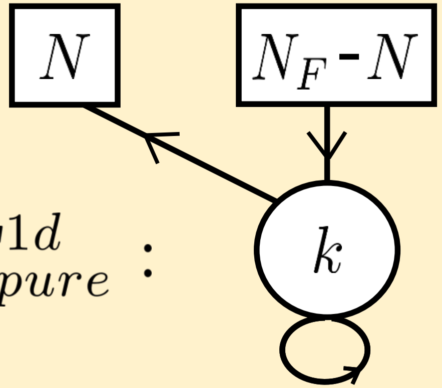


$$[\chi]^{(0)}_{1d} = \sum_{k=0}^{\infty} \frac{e^{\zeta_{3d}\, k}}{k!} \oint_{\mathcal{M}_k^{pure}} \left[\frac{d\phi_I}{2\pi i}\right] Z_{pure,vec} \cdot Z_{pure,adj} \cdot Z_{pure,teeth} \ ,$$

$$Z_{pure,vec} = \frac{\prod^k_{\substack{I\neq J\\ I,J=1}} \operatorname{sh}(\phi_I - \phi_J)}{\prod^k_{I,J=1} \operatorname{sh}(\phi_I - \phi_J + \epsilon_2)}$$

$$Z_{pure,adj} = \prod_{I,J=1}^{k} \frac{\operatorname{sh}(\phi_I - \phi_J + \epsilon_1 + \epsilon_2)}{\color{red}{\operatorname{sh}(\phi_I - \phi_J + \epsilon_1)}}$$

$$Z_{pure,teeth} = \prod_{I=1}^{k}\prod_{i=1}^{N} \frac{\operatorname{sh}(\phi_I - m_i + (\epsilon_1+\epsilon_2)/2)}{\color{red}{\operatorname{sh}(\phi_I - m_i + (\epsilon_1-\epsilon_2)/2)}} \prod_{j=N+1}^{N_f} \frac{\operatorname{sh}(-\phi_I + m_j + (\epsilon_1+3\epsilon_2)/2)}{\operatorname{sh}(-\phi_I + m_j + (\epsilon_1+\epsilon_2)/2)}$$

Though this is not required in our discussion, we note in passing that the closed form of the evaluated integrals is well-known:

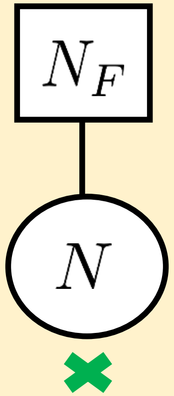
$$[\chi]^{(0)}_{1d} = \sum_{k=0}^{\infty} e^{\zeta_{3d}\, k} \sum_{\substack{\sum_i k_i = k \\ k_i \geq 0}} \left[\prod_{i,j=1}^{N} \prod_{s=1}^{k_i} \frac{\mathrm{sh}\left(m_i - m_j + \epsilon_2 - (s - k_j - 1)\,\epsilon_1\right)}{\mathrm{sh}\left(m_i - m_j - (s - k_j - 1)\,\epsilon_1\right)} \right]$$

$$\times \left[\prod_{i=N+1}^{N_F} \prod_{j=1}^{N} \prod_{p=1}^{k_j} \frac{\mathrm{sh}\left(m_i - m_j + \epsilon_2 + p\,\epsilon_1\right)}{\mathrm{sh}\left(m_i - m_j + p\,\epsilon_1\right)} \right]$$

The equivariant integrations over the based quasi map spaces is the equivariant J-function for the flag variety, which in Physics is the reduction on the circle of the index to two dimensions. The integral we presented here is the K-theoretic uplift [Givental-Lee '13].

We now add the defect:

3d Quiver G^{3d} :



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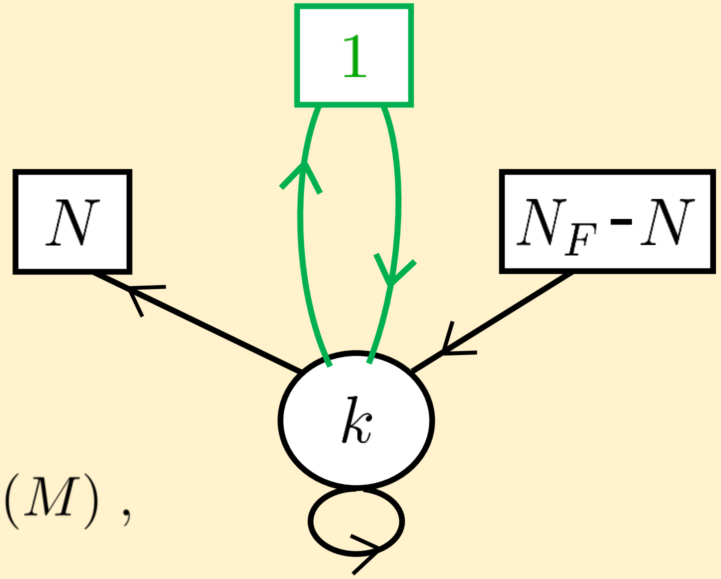

$$[\chi]^{(1)}_{1d}(M) = \sum_{k=0}^{\infty} \frac{e^{\zeta_{3d}\, k}}{k!} \oint_{\mathcal{M}_k} \left[\frac{d\phi_I}{2\pi i}\right] Z_{pure,vec} \cdot Z_{pure,adj} \cdot Z_{pure,teeth} \cdot Z_{defect}(M) \,,$$

$$Z_{pure,vec} = \frac{\prod_{\substack{I\neq J\\ I,J=1}}^{k} \mathrm{sh}\left(\phi_I - \phi_J\right)}{\prod_{I,J=1}^{k} \mathrm{sh}\left(\phi_I - \phi_J + \epsilon_2\right)}$$

$$Z_{pure,adj} = \prod_{I,J=1}^{k} \frac{\mathrm{sh}\left(\phi_I - \phi_J + \epsilon_1 + \epsilon_2\right)}{\mathrm{sh}\left(\phi_I - \phi_J + \epsilon_1\right)}$$

$$Z_{pure,teeth} = \prod_{I=1}^{k}\prod_{i=1}^{N} \frac{\mathrm{sh}\left(\phi_I - m_i + (\epsilon_1+\epsilon_2)/2\right)}{\mathrm{sh}\left(\phi_I - m_i + (\epsilon_1-\epsilon_2)/2\right)} \prod_{j=N+1}^{N_f} \frac{\mathrm{sh}\left(-\phi_I + m_j + (\epsilon_1+3\epsilon_2)/2\right)}{\mathrm{sh}\left(-\phi_I + m_j + (\epsilon_1+\epsilon_2)/2\right)}$$

$$Z_{defect}(M) = \prod_{I=1}^{k} \frac{\mathrm{sh}\left(\phi_I - M - (\epsilon_1-\epsilon_2)/2\right)\,\mathrm{sh}\left(-\phi_I + M - (\epsilon_1-\epsilon_2)/2\right)}{\mathrm{sh}\left(\phi_I - M - (\epsilon_1+\epsilon_2)/2\right)\,\mathrm{sh}\left(-\phi_I + M - (\epsilon_1+\epsilon_2)/2\right)} \,.$$

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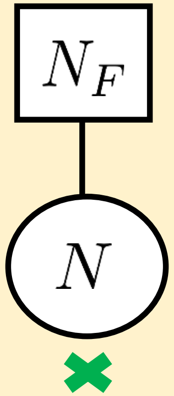

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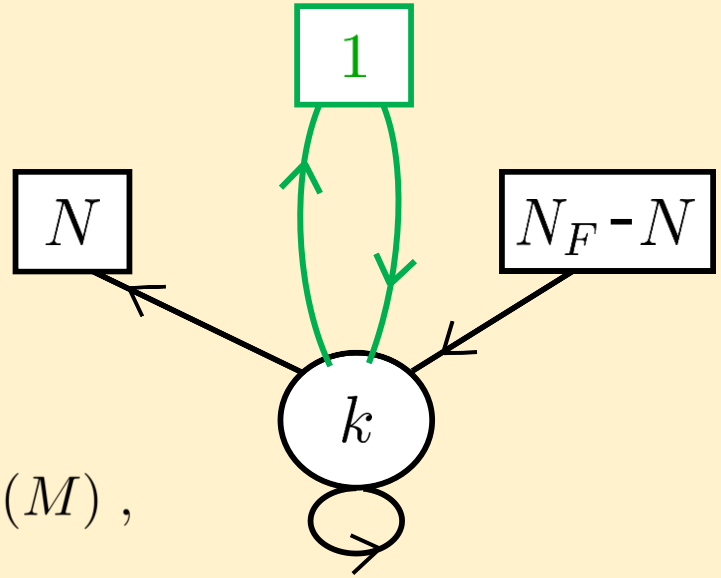

$$[\chi]^{(1)}_{1d}(M) = \sum_{k=0}^{\infty} \frac{e^{\zeta_{3d}\, k}}{k!} \oint_{\mathcal{M}_k} \left[\frac{d\phi_I}{2\pi i}\right] Z_{pure,vec} \cdot Z_{pure,adj} \cdot Z_{pure,teeth} \cdot Z_{defect}(M) \ ,$$

$$Z_{pure,vec} = \frac{\prod_{\substack{I \neq J \\ I,J=1}}^{k} \mathrm{sh}\left(\phi_I - \phi_J\right)}{\prod_{I,J=1}^{k} \mathrm{sh}\left(\phi_I - \phi_J + \epsilon_2\right)}$$

$$Z_{pure,adj} = \prod_{I,J=1}^{k} \frac{\mathrm{sh}\left(\phi_I - \phi_J + \epsilon_1 + \epsilon_2\right)}{\mathrm{sh}\left(\phi_I - \phi_J + \epsilon_1\right)}$$

$$Z_{pure,teeth} = \prod_{I=1}^{k}\prod_{i=1}^{N} \frac{\mathrm{sh}\left(\phi_I - m_i + (\epsilon_1 + \epsilon_2)/2\right)}{\mathrm{sh}\left(\phi_I - m_i + (\epsilon_1 - \epsilon_2)/2\right)} \prod_{j=N+1}^{N_f} \frac{\mathrm{sh}\left(-\phi_I + m_j + (\epsilon_1 + 3\epsilon_2)/2\right)}{\mathrm{sh}\left(-\phi_I + m_j + (\epsilon_1 + \epsilon_2)/2\right)}$$

$$Z_{defect}(M) = \prod_{I=1}^{k} \frac{\mathrm{sh}\left(\phi_I - M - (\epsilon_1 - \epsilon_2)/2\right) \mathrm{sh}\left(-\phi_I + M - (\epsilon_1 - \epsilon_2)/2\right)}{\mathrm{sh}\left(\phi_I - M - (\epsilon_1 + \epsilon_2)/2\right) \mathrm{sh}\left(-\phi_I + M - (\epsilon_1 + \epsilon_2)/2\right)} \ .$$

$$[\chi]^{(1)}_{1d} = \langle Y_{1d}(M) \rangle + e^{\zeta_{3d}} \prod_{i=1}^{N} \frac{\mathrm{sh}(M+\epsilon_1+\epsilon_2-m_i)}{\mathrm{sh}(M+\epsilon_1-m_i)} \prod_{j=N+1}^{N_F} \frac{\mathrm{sh}(m_j-M-\epsilon_2)}{\mathrm{sh}(m_j-M)} \left\langle \frac{1}{Y_{1d}(M+\epsilon_1+\epsilon_2)} \right\rangle$$

This is a twisted vortex character of the fundamental representation of the quantum affine algebra $U_q(\widehat{A_1})$.
The meaning of the above character is as follows: The first term on the right-hand side encloses almost all the “correct" poles in the index integrand, but it is missing exactly one: the extra pole at $\phi_I - M - \epsilon_+ = 0$. The second term on the right-hand side makes up for this missing pole, and relies on a key observation: we can trade a contour enclosing this extra pole for a contour which does not enclose it, at the expense of inserting the operator $Y_{1d}(M+\epsilon_1+\epsilon_2)^{-1}$ inside the vev. Notice the presence of the 3d F.I. parameter in the second term; it counts exactly one vortex, to make up for the missing M-pole, consistent with the fact that $|\mathcal{M}_k| = |\mathcal{M}_k^{pure}| + 1$.

$$[\chi]^{(1)}_{1d} = \langle Y_{1d}(M) \rangle + e^{\zeta_{3d}} \prod_{i=1}^{N} \frac{\mathrm{sh}(M+\epsilon_1+\epsilon_2-m_i)}{\mathrm{sh}(M+\epsilon_1-m_i)} \prod_{j=N+1}^{N_F} \frac{\mathrm{sh}(m_j-M-\epsilon_2)}{\mathrm{sh}(m_j-M)} \left\langle \frac{1}{Y_{1d}(M+\epsilon_1+\epsilon_2)} \right\rangle$$

The Schwinger-Dyson identities are a statement about the ‘on-shell’ regularity of the above character in the fugacity M. First, we note there are singularities in the fugacity M coming from the prefactors in the second term. Second, and more importantly, the vev $\langle Y_{1d}(M) \rangle$ is a priori also badly singular in the fugacity M. Nevertheless, and in a highly nontrivial fashion , it is the presence of the second term $\left\langle Y_{1d}(M+\epsilon_1+\epsilon_2)^{-1} \right\rangle$ cancels these apparent singularities.

All in all, the vortex character is only mildly singular in M , due to the chiral matter factors: this is the content of the Schwinger-Dyson identities for 3d $\mathcal{N}=4$ gauge theories.

Can we derive this result directly from the 3d picture, without resorting to the vortex quantum mechanics? [Pasquetti '11], [Krattenthaler-Spiridonov-Vartanov '11], [Dimofte-Gukov '11], [Beem-Dimofte-Pasquetti '12], [Hwang-Kim-Park '12], [Taki '13], [Cecotti-Gaiotto-Vafa '13], [Fujitsuka-Honda-Yoshida '13], [Benini-Peelaers '13], [Yoshida-Sugiyama '14], [Benini-Zaffaroni '15] …

Let us first review how to define an index for G^{3d} on $\mathbb{C} \times S^1(\widehat{R})$ in the absence of line defect. In Physics, this index is also referred to as a half-index, or holomorphic block. [Beem-Dimofte-Pasquetti '12]
We first consider the 3-manifold in the Ω-background, to regularize the non-compactness of $\mathbb{C}$; namely, if we let z_1 be a coordinate on the complex line, we can view the 3-manifold as a $\mathbb{C}$-bundle over $S^1(\widehat{R})$, where as we go around the circle, we make the identification:

$$z_1 \sim z_1 \, e^{\widehat{R}\,\epsilon_1} \,, \qquad \epsilon_1 \in \mathbb{R}$$

From now on, we denote the $\mathbb{C}$-line in this background as $\mathbb{C}_q$, with $q = e^{\widehat{R}\epsilon_1}$.
Then, the partition function of G^{3d} is defined via the following half-index:

$$[\chi]^{(0,\ldots,0)}_{3d} = \mathrm{Tr}\left[(-1)^F \, e^{-\widehat{R}\{Q,\overline{Q}\}} \, q^{S_1 - S_H} \, t^{-S_2+S_H} \prod_{a=1}^{n} \prod_{d=1}^{N_F^{(a)}} (x_d^{(a)})^{\Pi_d^{(a)}}\right]$$

As before, the index counts states in Q-cohomology. F is the fermion number. S_1 is a rotation generator for $\mathbb{C}_q$, while S_2 generates a $U(1)_C \subset SU(2)_C$ symmetry. Meanwhile, S_H generates a $U(1)_H \subset SU(2)_H$ symmetry. $\{\Pi_d^{(a)}\}$ are Cartan generators for the flavor group G_F , with conjugate fundamental masses $\{m_d^{(a)}\}$.

The gauge symmetry group $G=\prod_{a=1}^{n} U(N^{(a)})$ is first treated as a global symmetry, which we make abelian by breaking it to its maximal torus. One then gauges the symmetry by projecting to G-invariant states, which amounts to integrating over corresponding equivariant parameters:

$$[\chi]_{3d}^{(0,\ldots,0)} = \oint_{\mathcal{M}^{bulk}} dy \; \left[I_{bulk}^{3d}(y) \right]$$

The choice of contours $\mathcal{M}^{bulk}$ determines a vacuum for G^{3d}, and in particular determines the choice of a specific vortex quantum mechanics T^{1d}.
The integrand $I_{bulk}^{3d}(y)$ stands for the contribution of all the various multiplets to the index, which can be read off directly from the 3d $\mathcal{N}=4$ quiver:

$$I_{bulk}^{3d}(y) = \prod_{a=1}^{n} \prod_{i=1}^{N^{(a)}} y_i^{(a)\left(\zeta_{3d}^{(a)}-1\right)} \; I_{vec}^{(a)}(y) \cdot \prod_{b\neq a} I_{bif}^{(a,b)}(y) \cdot I_{flavor}^{(a)}(y, \{x_d^{(a)}\})$$

We now want to introduce a 1/2-BPS loop defect in G^{3d}. We couple the one-dimensional $\mathcal{N}=4$ theory on the loop to the bulk three-dimensional theory by considering the flavor symmetries of the 1d theory and gauging them with 3d vector multiplets. From the point of view of the index, this translates into gauging the 1d masses, turning them into the scalars of the corresponding 3d $\mathcal{N}=4$ vector multiplets. When the vector multiplet is dynamical, the scalar becomes an eigenvalue y to be integrated over, while in the case of a background vector multiplet, the scalar becomes a mass from the 3d point of view, and is not integrated over.

A defect on node a has the generic form:

$$\widetilde{Y}^{(a)}_{3d\ flavor/1d}(\{x^{(b)}_d\}, z) \cdot \oint_{\mathcal{M}^{bulk}} dy \left[I^{3d}_{bulk}(y) \cdot \widetilde{Y}^{(a)}_{3d\ gauge/1d}(y, z) \right]$$

The vortex character is then defined as a sum of such Coulomb branch localization integrals, with 3d/1d contributions included.
From the 3d perspective, the interpretation of the character as a sum over various integrals is rather mysterious: in the quantum mechanical picture, each term had an elegant interpretation as the residue of the vortex integrals at a defect-pole. No such interpretation is available here.

$$\widetilde{Y}^{(a)}_{3d\ flavor/1d}(\{x^{(b)}_d\}, z) \cdot \oint_{\mathcal{M}^{bulk}} dy\ \left[I^{3d}_{bulk}(y) \cdot \widetilde{Y}^{(a)}_{3d\ gauge/1d}(y, z)\right]$$

In 3d, the vortex character is defined instead by the requirement that all the z-singularities present in $\left\langle \widetilde{Y}^{(a)}_{3d\ gauge/1d}(y, z)\right\rangle$ should be cancelled out by other $\widetilde{Y}$-operator vevs; this construction is guaranteed to exist and be unique, from known facts about the representation theory of quantum affine algebras.

Example: SQCD.

In the absence of Wilson loop, the half-index of the 3d theory reads:

$$[\widetilde{\chi}]^{(0)}_{3d} = \oint_{\mathcal{M}^{bulk}} dy \, \left[I^{3d}_{bulk}(y) \right]$$

with the bulk contribution:

$$I^{3d}_{bulk}(y) = \prod_{i=1}^{N} {y_i}^{(\zeta_{3d}-1)} \, I_{vec}(y) \cdot I_{flavor}(y, \{x_d\})$$

$$I_{vec}(y) = \prod_{1 \le i \ne j \le N} \frac{(y_i/y_j;q)_\infty}{(t\,y_i/y_j;q)_\infty} \prod_{1 \le i < j \le N} \frac{\Theta\left(t\,y_i/y_j;q\right)}{\Theta\left(y_i/y_j;q\right)} \qquad (x\,;q)_\infty \equiv \prod_{s=0}^{\infty} (1 - q^s\,x)$$

$$I_{flavor}(y, \{x_d\}) = \prod_{d=1}^{N_F} \prod_{i=1}^{N} \frac{(t\,v\,x_d/y_i;q)_\infty}{(v\,x_d/y_i;q)_\infty} \qquad \Theta\left(x\,;q\right) \equiv (x\,;\,q)_\infty \, (q/x\,;\,q)_\infty$$

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$$I_{vec}(y) = \prod_{1 \le i \neq j \le N} \frac{(y_i/y_j;q)_\infty}{(t\,y_i/y_j;q)_\infty} \prod_{1 \le i < j \le N} \frac{\Theta\,(t\,y_i/y_j;q)}{\Theta\,(y_i/y_j;q)}$$

$$(x\,;q)_\infty \equiv \prod_{s=0}^{\infty} (1 - q^s\,x)$$

$$I_{flavor}(y, \{x_d\}) = \prod_{d=1}^{N_F} \prod_{i=1}^{N} \frac{(t\,v\,x_d/y_i;q)_\infty}{(v\,x_d/y_i;q)_\infty}$$

$$\Theta\,(x\,;q) \equiv (x\,;\,q)_\infty\ (q/x\,;\,q)_\infty$$

We now introduce the 1/2-BPS Wilson loop wrapping $S^1(\widehat{R})$ via gauging its 1d degrees of freedom. A corresponding defect $\widetilde{Y}$-operator vev is defined as an integral over the Coulomb moduli of the 3d theory:

$$\widetilde{Y}_{3d\ flavor/1d}(\{x_d\}, z) \cdot \oint_{\mathcal{M}^{bulk}} dy \left[I^{3d}_{bulk}(y) \cdot \widetilde{Y}_{3d\ gauge/1d}(y, z) \right]$$

with:

$$\widetilde{Y}_{3d\ gauge/1d}(y, z) = \prod_{i=1}^{N} \frac{1 - t\, y_i/z}{1 - y_i/z}$$

$$\widetilde{Y}_{3d\ flavor/1d}(\{x_d\}, z) = \prod_{d=1}^{N_F} \frac{1 - \sqrt{q/t}\, x_d/z}{1 - \sqrt{q\, t}\, x_d/z}$$

Then, the (normalized) vortex character of G^{3d} is *defined* as the following sum of two defect operator vevs:

$$[\chi]^{(1)}_{3d}(z) = \widetilde{Y}_{3d\ flavor/1d}(\{x_d\}, z) \left\langle \widetilde{Y}_{3d\ gauge/1d}(z) \right\rangle + e^{\zeta_{3d}} \left\langle \frac{1}{\widetilde{Y}_{3d\ gauge/1d}(z\,t/q)} \right\rangle$$

This is once again a twisted qq-character of the fundamental representation of the algebra $U_q(\widehat{A_1})$.

Up to overall normalization, this turns out to be the same character we had derived from the vortex quantum mechanics:

$$[\chi]^{(1)}_{1d} = \langle Y_{1d}(M) \rangle + e^{\zeta_{3d}} \prod_{i=1}^{N} \frac{\mathrm{sh}(M+\epsilon_1+\epsilon_2-m_i)}{\mathrm{sh}(M+\epsilon_1-m_i)} \prod_{j=N+1}^{N_F} \frac{\mathrm{sh}(m_j-M-\epsilon_2)}{\mathrm{sh}(m_j-M)} \left\langle \frac{1}{Y_{1d}(M+\epsilon_1+\epsilon_2)} \right\rangle$$

We will now see that the 3d vortex character observable has a very natural interpretation in the context of the BPS/CFT correspondence, as a correlator in deformed W-algebras.

The deformed W-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ is defined in Coulomb gas (i.e. free field) formalism. [Feigin-Frenkel '95] [Awata-Harunobu-Odake-Shiraishi '95]

[Awata-Harunobu-Odake-Shiraishi '95] [Frenkel-Reshetikhin '97]

In what follows, $\mathfrak{g}$ denotes a simply-laced Lie algebra.

The starting point is to define a deformed Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$, with generators

$$\alpha_a[k], \quad k \in \mathbb{Z}, \quad a = 1, \ldots, n$$

which have the following commutators:

$$[\alpha_a[k], \alpha_b[m]] = \frac{1}{k}(q^{\frac{k}{2}} - q^{-\frac{k}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})\widehat{C}_{ab}(q^{\frac{k}{2}}, t^{\frac{k}{2}})\delta_{k,-m}$$

(In the above, $\widehat{C}_{ab}(q^{\frac{k}{2}}, t^{\frac{k}{2}})$ is a modified Cartan matrix)

A Fock space representation of the Heisenberg algebra is constructed as follows: for each weight ψ of the Cartan subalgebra of $\mathfrak{g}$, the Fock representation is generated by a vector $|\psi\rangle$ such that

$$\alpha_a[0]|\psi\rangle = \langle\psi, \alpha_a\rangle|\psi\rangle$$
$$\alpha_a[k]|\psi\rangle = 0\,, \qquad \text{for } k > 0$$

The $\mathcal{W}_{q,t}(\mathfrak{g})$-algebra is defined as the associative algebra whose generators are Fourier modes of the operators commuting with the screening charges,

$$Q^{(a)} = \oint S^{(a)}(y)\,dy\ , \qquad S^{(a)}(y) = y^{-\alpha_a[0]}\ :\exp\Big(\sum_{k\neq 0} \frac{\alpha_a[k]}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}\,y^k\Big):$$

Namely, it is generated by n vertex operators $W^{(s)}(z)$ satisfying:

$$[W^{(s)}(z), Q^{(a)}] = 0\ , \qquad \text{for all } a = 1, \ldots, n, \text{ and } s = 2, \ldots, n+1\ .$$

These include a deformed Virasoro stress tensor and “higher spin” currents.

In this way, one finds that every generating current can be written as a Laurent polynomial in certain vertex operators, which we call Y-operators for reasons that will soon be clear:

$$\mathcal{Y}^{(a)}(z) = q^{w_a[0]} \ : \exp\left(-\sum_{k\neq 0} w_a[k]\, t^{-k/2}\, z^k \right) :$$

The modes $w_a[k]$ are obtained from the Heisenberg generators:

$$[\alpha_a[k], w_b[m]] = \frac{1}{k}(q^{\frac{k}{2}} - q^{-\frac{k}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})\, \delta_{ab}\, \delta_{k,-m} \qquad \alpha_a[k] = \sum_{b=1}^{n} C_{ab}(q^{\frac{k}{2}}, t^{\frac{k}{2}}) w_b[k]$$

Finally, we introduce n “fundamental” vertex operators:

$$V^{(a)}(x) = x^{w_a[0]} \ : \exp\left(-\sum_{k\neq 0} \frac{w_a[k]}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}\, x^k \right) :$$

We are interested in computing the following chiral correlators:

$$\left\langle \psi' \middle| \prod_{a=1}^{n} \prod_{d=1}^{N_f^{(a)}} V^{(a)}(x_d^{(a)}) \, (Q^{(a)})^{N^{(a)}} \prod_{s=2}^{n+1} \prod_{\rho=1}^{L^{(s-1)}} W^{(s)}(z_\rho^{(s-1)}) \middle| \psi \right\rangle ,$$

where we recall that the screening charges are integrals,

$$Q^{(a)} = \oint S^{(a)}(y)\, dy \ ,$$

so one also needs to specify the contour. This can be done explicitly, using again the Jeffrey-Kirwan prescription.

Because of the free field formalism, the evaluation of the correlator can be done in a straightforward way using Wick contractions.

Up to overall normalization, we obtain the following result:

$$\left\langle \psi' \middle| \prod_{a=1}^{n} \prod_{d=1}^{N_f^{(a)}} V^{(a)}(x_d^{(a)}) \, (Q^{(a)})^{N^{(a)}} \prod_{s=2}^{n+1} \prod_{\rho=1}^{L^{(s-1)}} W^{(s)}(z_\rho^{(s-1)}) \middle| \psi \right\rangle = [\chi]_{3d}^{(L^{(1)},\ldots,L^{(n)})} (\{z_\rho^{(s-1)}\})$$

Namely, the correlator is the vortex character of the 3d gauge theory G^{3d} !

Recall that the Schwinger-Dyson identity was a statement about the on-shell regularity of the vortex character in the fugacities $\{z_\rho^{(s-1)}\}$. It is reinterpreted here as a Ward identity for a correlator involving fundamental vertex operator in the deformed $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra.

For example, in deformed Liouville $\mathcal{W}_{q,t}(A_1)$, the generating current has the form:

$$W^{(2)}(z) = \mathcal{Y}(z) + [\mathcal{Y}(z\,t/q)]^{-1} \ .$$

As an aside, note that one can turn off the deformation by taking the limit:

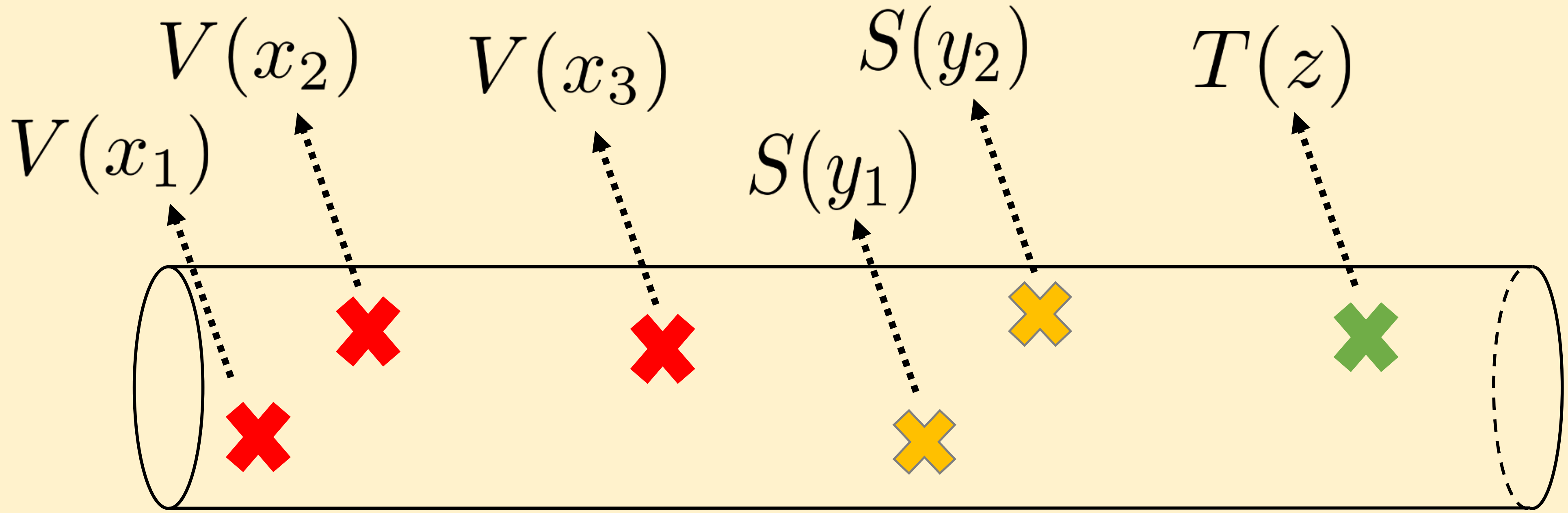
$$t = q^\gamma \, , \qquad q \to 1$$

This limit takes $\mathcal{W}_{q,t}(\mathfrak{g})$ to the undeformed algebra $\mathcal{W}_\gamma(\mathfrak{g})$, where the central charge is determined from γ . In particular, we recover the well-known Virasoro stress tensor in the limit:

$$W^{(2)}(z) \qquad \longrightarrow \qquad -\frac{1}{2} : (\partial_z \phi(z))^2 : + Q \ : \partial_z^2 \phi(z) :$$

In this A_1 example, the correlator of interest is:

$$\left\langle \psi' \middle| \prod_{d=1}^{N_f} V(x_d) \prod_{i=1}^{N} \oint dy_i \, S(y_i) \, W^{(2)}(z) \middle| \psi \right\rangle .$$



We recognize the vortex character of the 3d gauge theory:

$$\left\langle \psi' \middle| \prod_{d=1}^{N_f} V(x_d) \prod_{i=1}^{N} \oint dy_i\, S(y_i)\; W^{(2)}(z) \middle| \psi \right\rangle .$$

$$[\chi]^{(1)}_{3d}(z) = \left[\widetilde{Y}_{3d\ flavor/1d}(\{x_d\}, z) \left\langle \widetilde{Y}_{3d\ gauge/1d}(z) \right\rangle + e^{\zeta_{3d}} \left\langle \frac{1}{\widetilde{Y}_{3d\ gauge/1d}(z\,t/q)} \right\rangle \right]$$

$$\left\langle V(x_d)\, W^{(2)}(z) \right\rangle \left\langle S(y_i)\, W^{(2)}(z) \right\rangle$$

$${y_i}^{\langle \psi, \alpha \rangle} \qquad \langle S(y_i)\, S(y_j) \rangle \qquad \langle V(x_d)\, S(y_i) \rangle$$

$$\left\langle \widetilde{Y}^{\pm 1}_{3d\ gauge/1d}(z) \right\rangle = \oint_{\mathcal{M}^{bulk}} dy_i \prod_{i=1}^{N} {y_i}^{(\zeta_{3d}-1)}\; I_{vec}(y) \cdot I_{flavor}(y, \{x_d\}) \cdot \left[\widetilde{Y}_{3d\ gauge/1d}(y, z) \right]^{\pm 1}$$

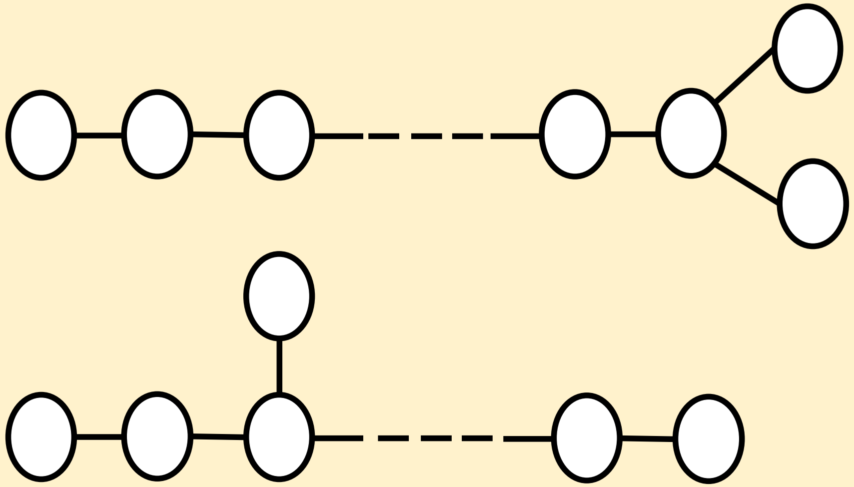
In this last part of the talk, we give a string theory realization of the vortex character observable, and will explain how it is related to our previous gauge theory and W-algebra descriptions.

For definiteness, consider:

$$\text{Type IIB on } \widetilde{\mathbb{C}^2/\Gamma_{\mathfrak{g}}} \times \mathcal{C} \times \mathbb{C}^2$$

$\mathcal{C} = S^1(R) \times \mathbb{R}$ is an infinite cylinder of radius R.

$\widetilde{\mathbb{C}^2/\Gamma_{\mathfrak{g}}}$ is a resolved ADE singularity, labeled by a discrete subgroup of $SU(2)$:

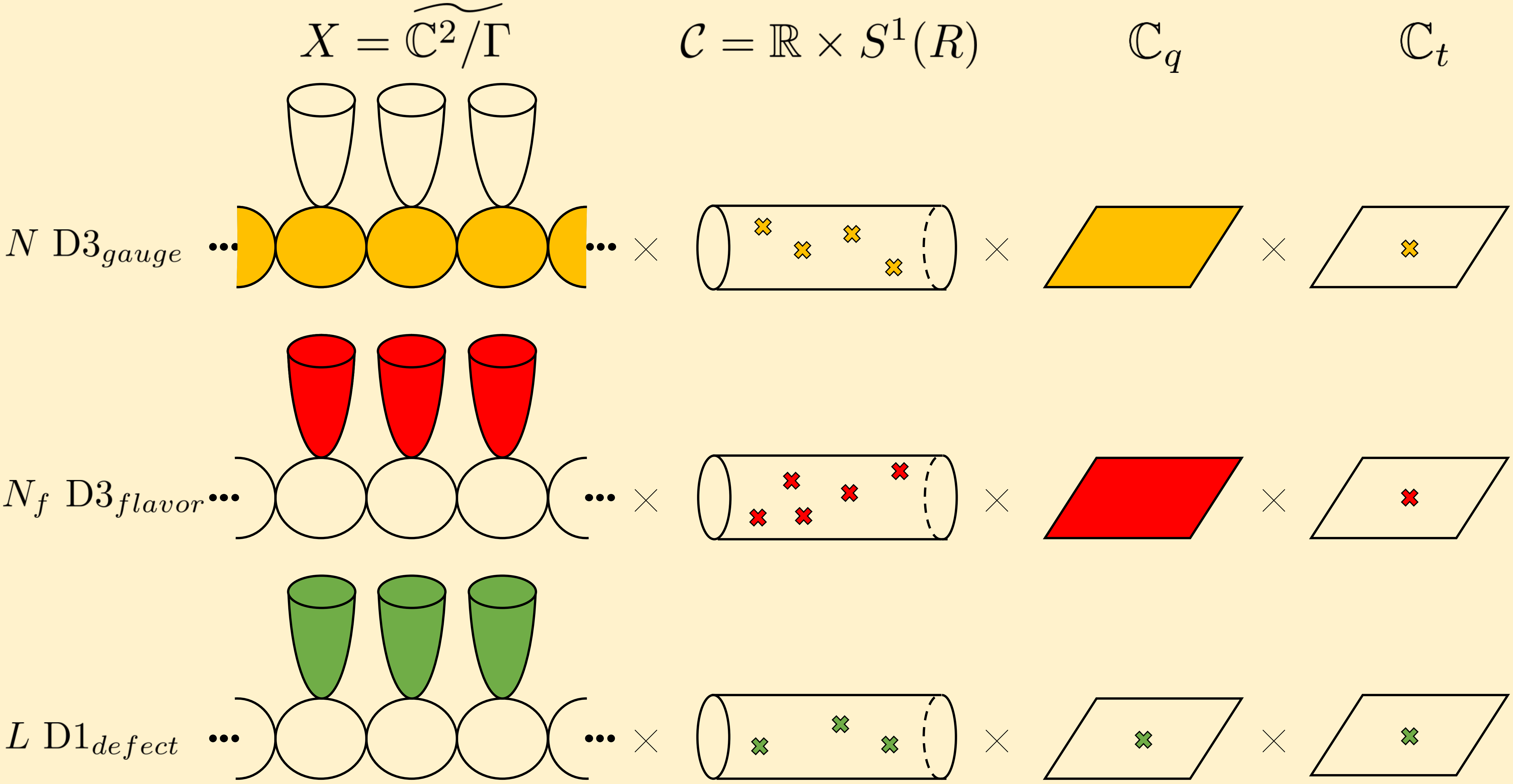


A_n

D_n

E_6, E_7, E_8

So far the background has 16 supercharges. Since we are ultimately interested in the dynamics of three-dimensional gauge theories with 8 supercharges, we need to break supersymmetry further. A simple way to achieve this in string theory is to add D-branes.



$X = \widetilde{\mathbb{C}^2/\Gamma}$
$\mathcal{C} = \mathbb{R} \times S^1(R)$
$\mathbb{C}_q$
$\mathbb{C}_t$
N D3$_{gauge}$
N_f D3$_{flavor}$
L D1$_{defect}$

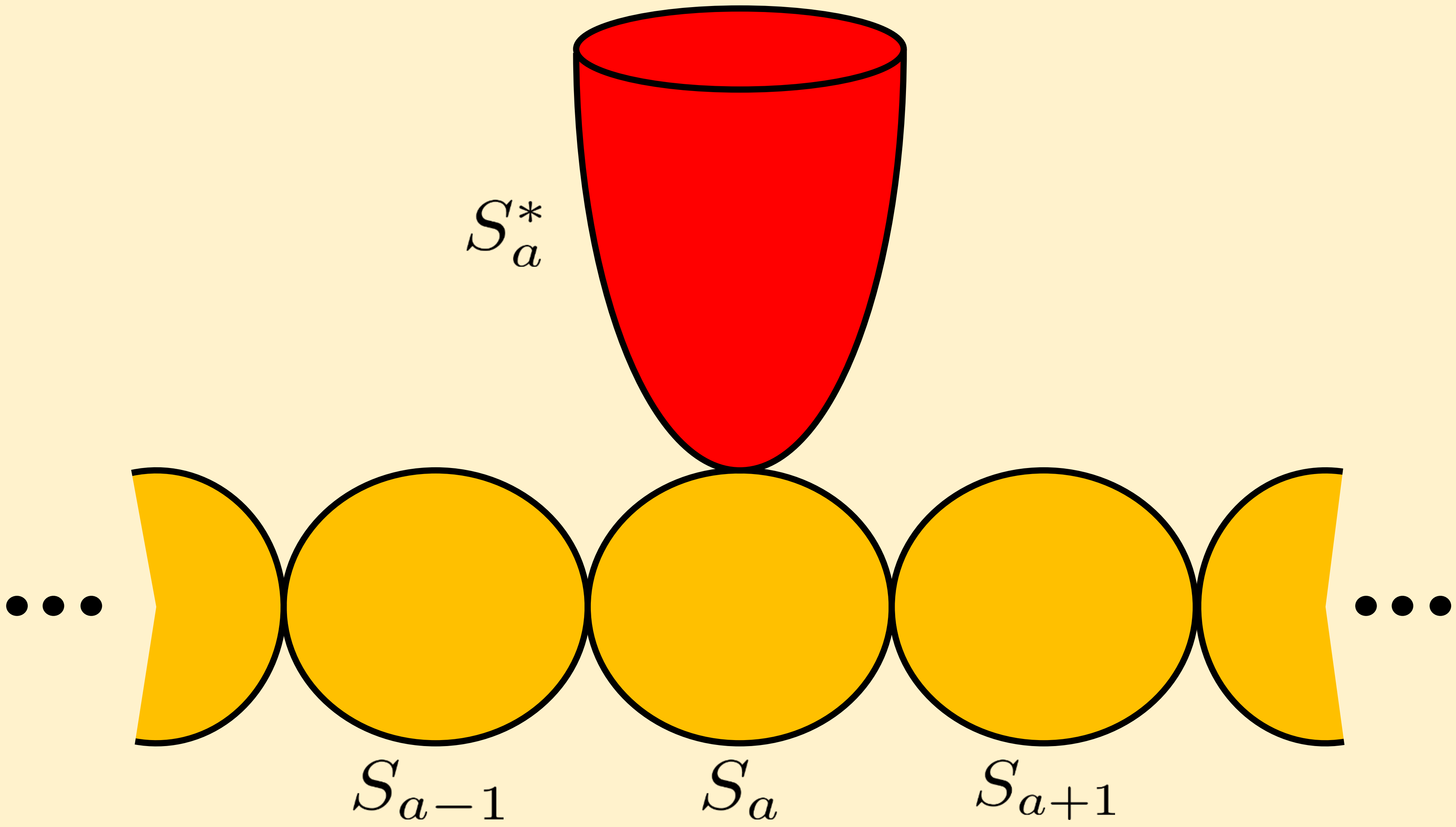
By the McKay Correspondence, we can choose a class in $H_2(X;\mathbb{Z})$ to represent the brane charge due to the D3 branes wrapping the compact two-cycles, expanded in terms of positive simple roots:

$$[S] = -\sum_{a=1}^{n} N^{(a)}\,\alpha_a \qquad\qquad (\,[S_a] \equiv \alpha_a\,)$$

Likewise, we choose a class in $H_2(X,\partial X;\mathbb{Z})$ to represent the total charge due to the D3 branes wrapping the noncompact two-cycles, expanded in terms of fundamental weights:

$$[S^*] = \sum_{a=1}^{n} N_F^{(a)}\,\lambda_a \qquad\qquad (\,[S_a^*] \equiv \lambda_a\,)$$

$$\#(S_a \cap S_b^*) = \delta_{ab}$$


S_a^*
S_{a-1}
S_a
S_{a+1}

Finally, we choose a class in $H_2(X, \partial X; \mathbb{Z})$ to represent the total charge due to the D1 branes wrapping the noncompact two-cycles, expanded in terms of fundamental weights:

$$[L] = \sum_{a=1}^{n} L^{(a)} \, \lambda_a$$

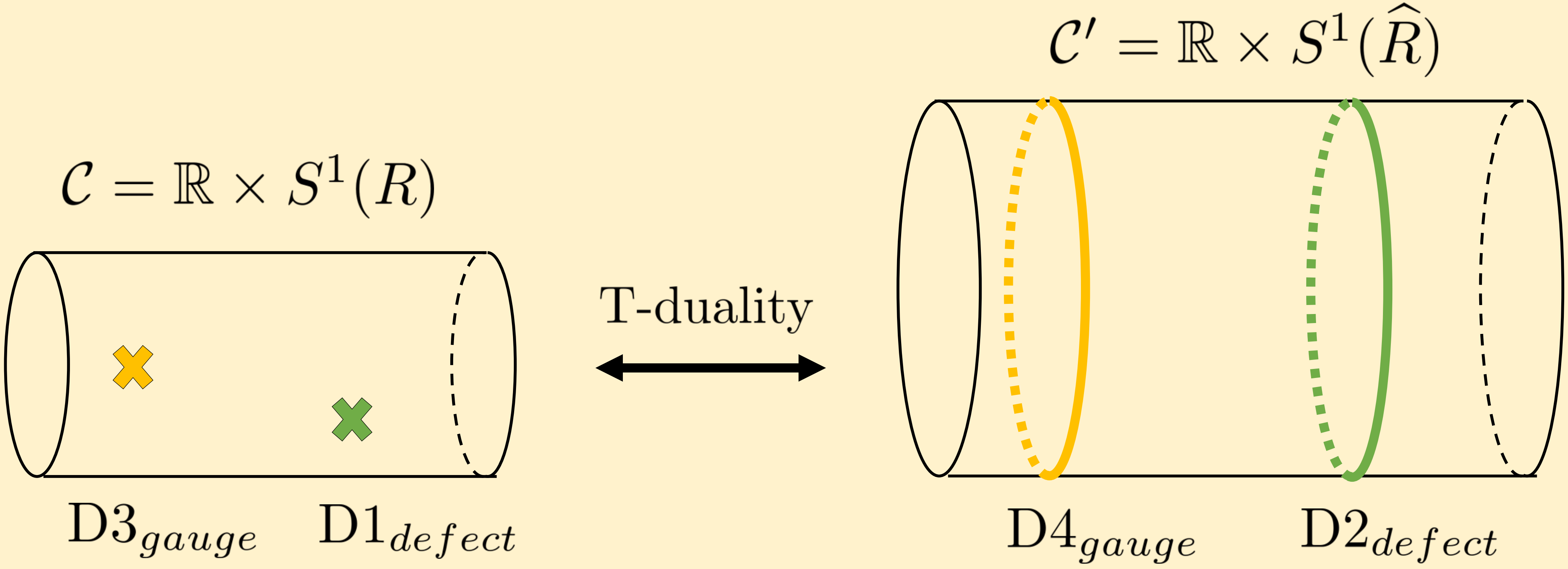
We will only focus on the degrees of freedom supported near the singularity at the origin of X, and decouple gravity. This is achieved in string theory by taking the string coupling limit $g_s \to 0$.

In the limit, the resulting 6-dimensional theory on $\mathcal{C} \times \mathbb{C}^2$ is a (noncritical) theory of strings known as the (2,0) little string theory.

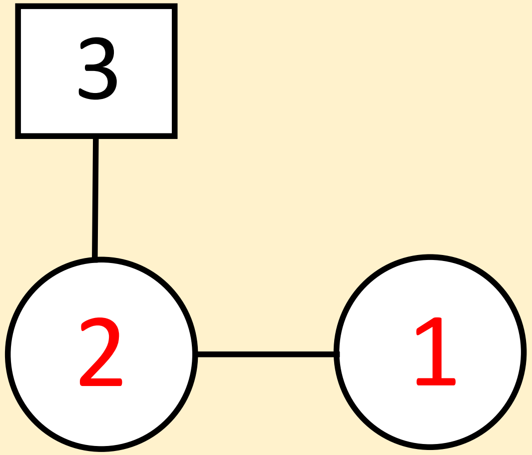
The low energy theory on the D3 branes is a quiver gauge theory, of shape the Dynkin diagram of $\mathfrak{g}$. [Douglas-Moore '96]

The low energy theory on the D3 branes has the Poincare invariance of a two-dimensional $\mathcal{N}=(4,4)$ gauged linear sigma model, but it really is a 3d N=4 theory, on the manifold $\mathbb{C}\times S^1(\widehat{R})$, where $\widehat{R}$ is T-dual to R. This is precisely the theory we have been calling G^{3d}.

Likewise, the D1 branes are really loops wrapping $S^1(\widehat{R})$. In a nontrivial B-field, these are ½-BPS defects of G^{3d}.



Example of G^{3d} when $\mathfrak{g} = A_2$:



$$[S^*] = 3\,[S_1^*]$$

$$[S] = 2\,[S_1] + [S_2]$$

Positions of non-compact D3 branes on C are mass parameters of the low energy quiver gauge theory G^{3d}.

Positions of compact D3 branes on C are Coulomb moduli of G^{3d}.

The gauge couplings and F.I. parameters come from various moduli of the metric on $X = \widetilde{\mathbb{C}^2/\Gamma}$ and spacetime NS-NS and R-R B-fields:

$$\frac{m_s^4}{g_s}\int_{S_a}\omega_{I,J,K}\,, \qquad \frac{m_s^2}{g_s}\int_{S_a}B^{(2)}\,, \qquad m_s^2\int_{S_a}C^{(2)}$$

Positions of the D1 branes on C are the masses for 1d $\mathcal{N}=4$ chiral multiplets; these arise from quantizing the D1/D3 strings.

We want to compute the index of the (2,0) little string theory on $\mathcal{C} \times \mathbb{C}_q \times \mathbb{C}_t$ in the presence of the various branes. This is equivalent to computing the index of the (1,1) little string on $\mathcal{C}' \times \mathbb{C}_q \times \mathbb{C}_t$, by T-duality. The latter picture makes the connection to the beginning of the talk explicit.

Indeed, by a supersymmetric localization argument, the index of the little string becomes the index of the theory on the defects. When only $\mathrm{D4}_{gauge}$ and $\mathrm{D4}_{flavor}$ branes are present, this computes the partition function of the gauge theory G^{3d}.

We now introduce the $\mathrm{D2}_{defect}$ branes; these branes are nondynamical as they do not wrap $\mathbb{C}_q$, but they nonetheless modify the index by introducing a Y-operator insertion:

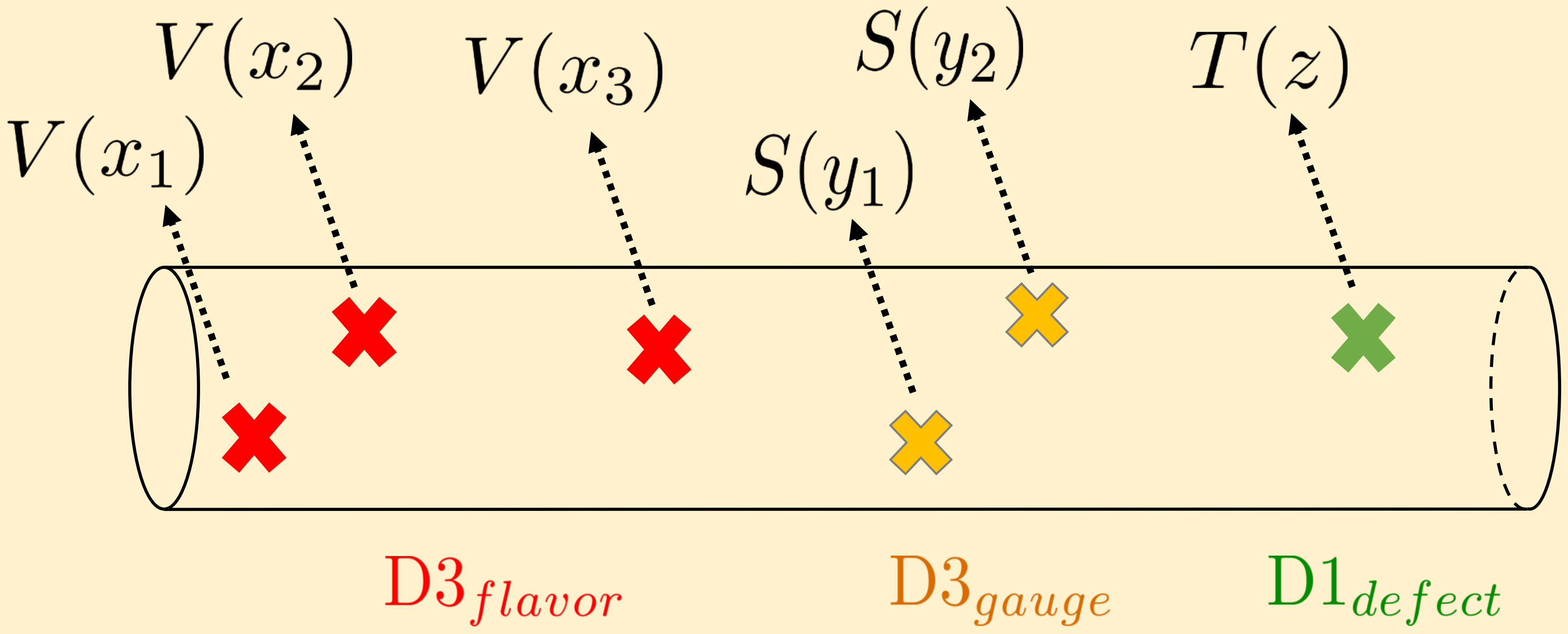
$$\widetilde{Y}_{3d\ flavor/1d}(\{x_d\}, z) \cdot \oint_{\mathcal{M}^{bulk}} dy \left[I^{3d}_{bulk}(y) \cdot \widetilde{Y}_{3d\ gauge/1d}(y, z) \right]$$

Chiral multiplets $\widetilde{Y}_{3d\ flavor/1d}(\{x_d\}, z)$ are provided by $\mathrm{D4}_{flavor}/\mathrm{D2}_{defect}$ strings, while chiral multiplets $\widetilde{Y}_{3d\ gauge/1d}(y, z)$ are provided by $\mathrm{D4}_{gauge}/\mathrm{D2}_{defect}$ strings.

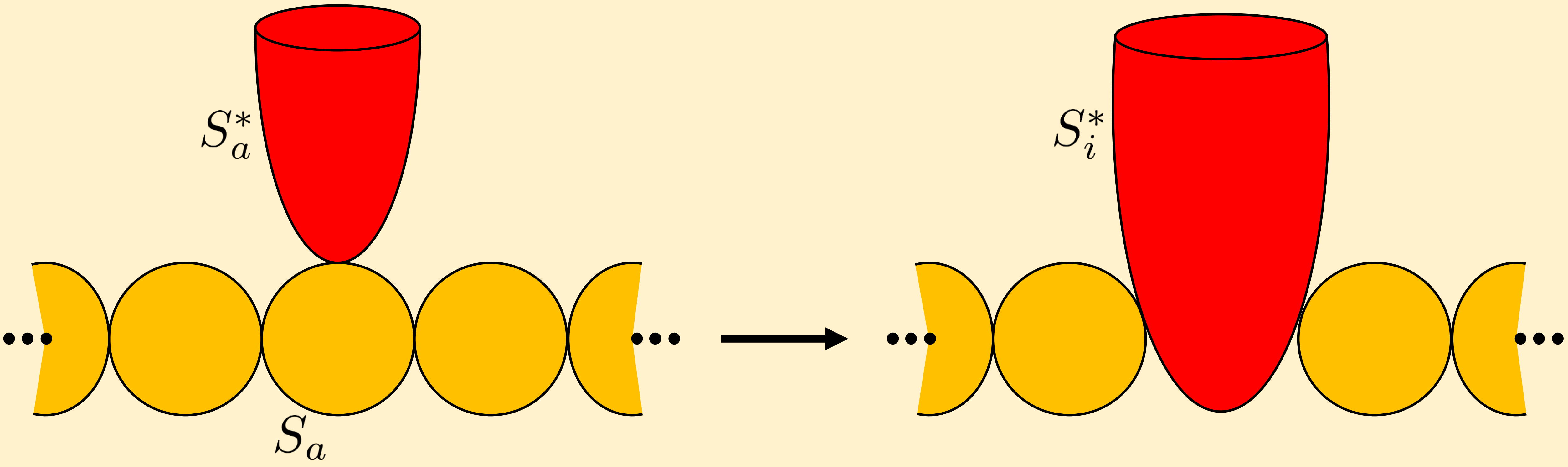
All in all, this implies that the (normalized) index of the little string in the presence of all three types of branes localizes to the vortex qq -character observable:

$$[\widetilde{\chi}]^{(L^{(1)},\ldots,L^{(n)})}_{\mathrm{D4}_{gauge}/\mathrm{D4}_{flavor}/\mathrm{D2}_{defect}}\left(\{z^{(a)}_{\rho}\}\right) = [\chi]^{(L^{(1)},\ldots,L^{(n)})}_{3d}\left(\{z^{(a)}_{\rho}\}\right)$$

The above result also makes the dictionary to deformed $\mathcal{W}_{q,t}(\mathfrak{g})$-algebras explicit: the screening charges are the $\mathrm{D3}_{gauge}$ branes, the fundamental vertex operators are the $\mathrm{D3}_{flavor}$ branes, and the generating currents are the $\mathrm{D1}_{defect}$ branes. The little string index can therefore be recast as a $\mathcal{W}_{q,t}(\mathfrak{g})$-algebra correlator:



To make contact with the vortex quantum mechanics T^{1d}, we need to do a little more work. Namely, we freeze the moduli of the $\mathrm{D3}_{gauge}$ branes to be equal to the moduli of the $\mathrm{D3}_{flavor}$ branes. This describes the root of the Higgs branch for the theory G^{3d} . Geometrically, this means we can recombine the $\mathrm{D3}_{gauge}$ branes with the $\mathrm{D3}_{flavor}$ branes so that they exclusively make up a collection of $\mathrm{D3'}_{flavor}$ branes wrapping the non-compact 2-cycles of X, and the theory is effectively massive:



Now, we would like to introduce vortices for G^{3d} . We force the theory on the Higgs branch by turning on the periods $\int_{S_a} \omega_I > 0$, which breaks the $SU(2)_H$ R-symmetry of the little string to $U(1)_H$.

Correspondingly, this turns on n real F.I. parameter for G^{3d}. The vortices are realized as $\mathrm{D1}_{vortex}$ branes wrapping the compact 2-cycles of X, at a point on the cylinder $\mathcal{C}$ in the (2,0) little string.

Alternatively, they are $\mathrm{D2}_{vortex}$ branes wrapping the compact 2-cycles of X and the circle of the cylinder $\mathcal{C}'$ in the (1,1) little string.

All in all, this implies that the (normalized) index of the little string in the presence of all three types of branes localizes to the vortex qq -character observable:

$$[\widetilde{\chi}]^{(L^{(1)},\ldots,L^{(n)})}_{\mathrm{D2}_{vortex}/\mathrm{D4'}_{flavor}/\mathrm{D2}_{defect}}\left(\{z^{(a)}_{\rho}\}\right) = [\chi]^{(L^{(1)},\ldots,L^{(n)})}_{1d}\left(\{z^{(a)}_{\rho}\}\right)$$

In this picture, recall that the index depends on a choice of sign for the 1d F.I. parameters, and experiences wall crossing. In the little string context, this is the sign of the periods $\int_{S_a} B^{(2)}$.

In conclusion, Schwinger-Dyson identities pertaining to the change in vortex number of a 3d $\mathcal{N}=4$ theory (on a circle) can be recast in the analytic properties of an observable we called the vortex character, with quantum affine symmetry. We gave various physical realizations of it:

-- The vortex character is the Witten index of a one-dimensional gauged quantum mechanics living on the vortices of G^{3d}, with additional chiral matter due to the defect.

-- The vortex character is a sum of half-indices for G^{3d}, in the presence of a codimension-2 defect. More precisely, each term in the sum is a 3d/1d half-index, where one-dimensional degrees of freedom due to the defect are coupled to the bulk 3d theory.

-- The vortex character is a deformed $\mathcal{W}_{q,t}(\mathfrak{g})$-algebra correlator on an infinite cylinder, with stress tensor and higher spin current insertions, and a distinguished set of ‘fundamental’ vertex operators.

-- The vortex character is the index of the six-dimensional (2,0) little string theory compactified on the cylinder, in the presence of codimension-4 and point-like defects; these are various D branes of type IIB on a resolved ADE singularity.

Future directions:

There are immediate generalizations in many directions:

- Beyond the ADE quivers
- Beyond the unitary gauge groups
- Vortices of 4d theories on a torus (quantum elliptic algebras)
- Schwinger-Dyson Physics of 3d Theories with less supersymmetry…

An important open question: what is the meaning of the vortex character and the associated Schwinger-Dyson equations in quantum K-theory?

Thank you!